11

Algebra

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THE BEGINNING OF ALGEBRA: AL-KHWĀRIZMĪ

The 'publication' of the book of al-Khwārizmī at the beginning of the ninth century – between 813 and 833[1] – is an outstanding event in the history of mathematics. For the first time, one could see the term algebra appearing in a title[2] to designate a distinct mathematical discipline, equipped with a proper technical vocabulary. Muḥammad ibn Mūsā al-Khwārizmī, mathematician, astronomer and distinguished member of the 'House of Wisdom' of Baghdad, had compiled, he wrote, 'a book on algebra and *al-muqābala*, a concise book recording that which is subtle and important in calculation'.[3] The event was crucial, and was recognized as such by both ancient and modern historians. Its importance did not escape the mathematical community of the epoch,[4] nor that of the following centuries. This book of al-Khwārizmī did not cease being a source of inspiration and the subject of commentaries by mathematicians, not only in Arabic and Persian, but also in Latin and in the languages of Western Europe until the eighteenth century. But the event appeared paradoxical: to the novelty of the conception, of the vocabulary and of the organization of the book of al-Khwārizmī was contrasted the simplicity of the mathematical techniques described, if one compares them with the techniques in the celebrated mathematical compositions, of Euclid or Diophantus, for example. But this technical simplicity stems precisely from the new mathematical conception of al-Khwārizmī. Whilst one of the elements of his project was found twenty-five centuries before him with the Babylonians, another in the *Elements* of Euclid, a third in the *Arithmetica* of Diophantus, no earlier writer had recompiled these elements, and in this manner. But which are these elements, and what is this organization?

The goal of al-Khwārizmī is clear, never conceived of before: to elaborate a theory of equations solvable through radicals, which can be applied to whatever arithmetical and geometrical problems, and which can help in calculation, commercial transactions, inheritance, the surveying of land etc.

In the first part of his book, al-Khwārizmī begins by defining the basic terms of this theory which, because of the requirement of resolution by radicals and because of his know-how in this area, was only concerned with equations of the first two degrees. In fact it is about the unknown, casually denoted by *root* or *thing*, its square, rational positive numbers, the laws of arithmetic $\pm$, $\times / \div$, $\sqrt{}$, and equality. The principal concepts introduced next by al-Khwārizmī are the equation of the first degree, the equation of the second degree, the binomials and the associated trinomials, the normal form, algorithmic solutions, and the demonstration of the solution formula. The concept of equation appeared in the book of al-Khwārizmī to designate an infinite class of problems, and not, as with the Babylonians for example, in the course of the solution of one or other problem. However, the equations are not presented in the course of the solution of problems to solve, like the ones of the Babylonians and Diophantus, but from the start, from the basic terms whose combinations must give all the possible forms. Thus, al-Khwārizmī, immediately after having introduced the basic terms, gives the six following types:

$$ax^2 = bx \qquad ax^2 = c \qquad bx = c$$

$$ax^2 + bx = c \qquad ax^2 + c = bx \qquad ax^2 = bx + c$$

He then introduces the notion of normal form, and needs to reduce each of the preceding equations to the corresponding normal form. He finds in particular, for the trinomial equations,

$$x^2 + px = q \qquad x^2 = px + q \qquad x^2 + q = px \tag{1}$$

Al-Khwārizmī next passed to the determination of algorithmic formulae for the solutions. He treated each case, and obtained formulae equivalent to the following expressions:

$$x = \left[\left(\frac{p}{2}\right)^2 + q\right]^{1/2} - \frac{p}{2}$$

$$x = \frac{p}{2} + \left[\left(\frac{p}{2}\right)^2 + q\right]^{1/2}$$

$$x = \frac{p}{2} \pm \left[\left(\frac{p}{2}\right)^2 - q\right]^{1/2} \qquad \text{if } \left(\frac{p}{2}\right)^2 > q$$

and in this last case he clarifies[5]

if $\left(\frac{p}{2}\right)^2 = q$ 'then the root of the square [*māl*] is equal to half of the number of roots, exactly, without surplus or diminution' if $\left(\frac{p}{2}\right)^2 < q$ 'then the problem is impossible'

Al-Khwārizmī also demonstrates different formulae, not algebraically, but by means of the idea of equality of areas. He was probably inspired by a very recent knowledge of the *Elements* of Euclid, translated by his colleague at the House of Wisdom, al-Ḥajjāj ibn Maṭar. Each of these demonstrations is presented by al-Khwārizmī as the 'cause' – *ʿilla* – of the solution. Also al-Khwārizmī not only required each case to be demonstrated, but he proposed sometimes two demonstrations for one and the same type of equation. One such requirement marks well the distance covered, and not only separates al-Khwārizmī from the Babylonians, but also, by his systematic working from now on, from Diophantus.

Thus, for example, for the equation $x^2 + px = q$, he takes two segments AB = AC = x and then takes CD = BE = $p/2$ (Figure 11.1). If the sum of the surfaces ABMC, BENM, DCMP is equal to q, the surface of the square AEOD is equal to $(p/2)^2 + q$, whence[6]

$$x = \left[\left(\frac{p}{2}\right)^2 + q\right]^{1/2} - \frac{p}{2}$$

With al-Khwārizmī, the concepts of the new discipline, and notably 'the thing', the unknown, are not designated to be a particular entity but an object which can be either numerical or geometrical; on the other hand the algorithms of the solution must be themselves an object of demonstration. It is there that the principal elements of the contribution of al-Khwārizmī reside. As he saw it, all problems dealt with from now on in algebra,

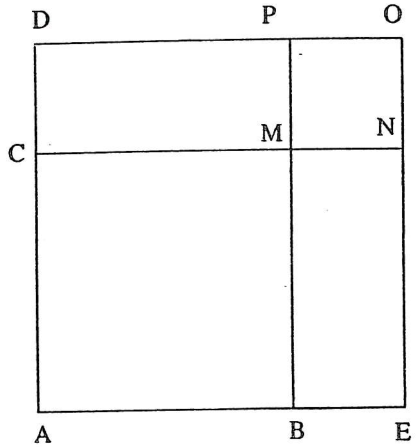

Figure 11.1

whether they be arithmetic or geometry, must be reduced to an equation with a single unknown and with positive rational coefficients of second degree at most. The algebraic operations – transposition and reduction – are then applied to put the equation in normal form, which makes possible the idea of a solution as a simple procedure of decision, an algorithm for each class of problems. The formula of the solution is then justified mathematically, with the help of a proto-geometric demonstration, and al-Khwārizmī is in a position to write that everything found in algebra 'must lead you to one of the six types that I described in my book'.[7]

Al-Khwārizmī then undertakes a brief study of some properties of the application of elementary laws of arithmetic to the simplest algebraic expressions. He studies in this way products of the type

$$(a \pm bx)(c \pm dx) \qquad \text{with } a, b, c, d \in Q_+$$

As rudimentary as it appears to be, this study represents no less than the first attempt at algebraic calculation as such, since the elements of this calculation became the subject of relatively autonomous chapters. These are then followed by other chapters in which al-Khwārizmī proceeds to the application of a worked out theory, in order to solve numerical and geometrical problems, before treating at last the problems of inheritance with the aid of algebra, in which he comes across some problems of indeterminate analysis.

Thus, at first, algebra is presented as a kind of arithmetic, more general than the 'logistic' – because it allows 'logistic' problems to be solved more rigorously thanks to these concepts – but also more general than metric geometry. The new discipline is in fact a theory of linear and quadratic equations with a single unknown solvable by radicals, and of algebraic calculation on the associated expressions, without yet the concept of a polynomial.

THE SUCCESSORS OF AL-KHWĀRIZMĪ AND THE DEVELOPMENT OF ALGEBRAIC CALCULATION

In order to grasp better the idea that al-Khwārizmī developed in the new discipline, as well as its fruitfulness, it is certainly insufficient to compare his book with ancient mathematical compositions; it is also necessary to examine the impact that he had on his contemporaries and on his successors. It is only then that he rises up in his true historical dimension. One of the features of this book, essential to our minds, is that it immediately aroused a trend of algebraic research. The biobibliographer of the tenth century, al-Nadīm, has delivered us already a long list of contemporaries and of successors of al-Khwārizmī who followed his research. Amongst many others were Ibn Turk, Sind ibn ʿAlī, al-Ṣaydanānī, Thābit ibn Qurra, Abū Kāmil, Sinān ibn al-Fatḥ, al-Ḥubūbī and Abū al-Wafā' al-Būzjānī. Although a good number of their writings have disappeared, enough have reached us to restore the main lines of this tradition, but it is not possible for us within the limits of this chapter to take up an analysis of each of the contributions. We attempt only to extract the principal axes of the development of algebra following al-Khwārizmī.

In the time of al-Khwārizmī and immediately afterwards, we witness essentially the expansion of research already begun by him: the theory of quadratic equations, algebraic calculation, indeterminate analysis and the application of algebra to problems of inheritance, partition etc. Research into the theory of equations was down several avenues. The first was that already opened up by al-Khwārizmī himself, but this time with an improvement of his proto-geometric demonstrations: it is the path followed by Ibn Turk[8] who, without adding anything new, reproduced a tighter discussion of the proof. More important is the path that Thābit ibn Qurra took a little later. He comes back to the *Elements* of Euclid, both to establish the demonstrations of al-Khwārizmī on more solid geometrical bases and to explain equations of second degree geometrically. Moreover, Ibn Qurra is the first to distinguish clearly between the two methods, algebraic and geometrical, and he seeks to show that they both lead to the same result, i.e. to a geometrical interpretation of algebraic procedures. Ibn Qurra begins by showing that the equation $x^2 + px = q$ can be solved with the help of proposition II.6 of the *Elements*. At the end of his proof, he writes: 'this method corresponds to the method of the algebraists – *aṣḥāb al-jabr*'.[9] He continues with $x^2 + q = px$ and $x^2 = px + q$, with the help respectively of II.5 and II.6 of the *Elements*; he shows for each the correspondence with the algebraic solutions, and writes: 'The method for solving this problem and the one that precedes it by geometry is the way of its solution by algebra'.[10] The mathematicians subsequently confirmed these conclusions. One of them writes: 'It was shown that the procedure which led to the determination of the sides of the unknown squares in each of three trinomial equations is the procedure given by Euclid at the end of the sixth book of his work on the *Elements*, and which is to apply to a given straight line a parallelogram which exceeds the whole parallelogram or which is deficient by a square. The side of the square in excess is the side of the unknown square in the first and second trinomial equations ($x^2 + q = px$, $x^2 + px = q$), and in the third trinomial equation it is the sum of the straight line on which the parallelogram is applied and the side of the square in excess'.[11]

But this geometrical explanation by Ibn Qurra of the equations of al-Khwārizmī proved to be particularly important, as we shall see, in the development of the theory of algebraic equations. Another account, very different, appeared at nearly the same time, and it also would be fundamental for the development of the theory of algebraic equations: the explanation of geometrical problems in algebraic terms. Indeed al-Māhānī, a contemporary of Ibn Qurra, began not just the translation of certain biquadratic problems in book X of the *Elements* into algebraic equations, but also a problem on solids, given in Archimedes' *The Sphere and the Cylinder*, in a cubic equation.[12]

Another direction of development of the theory of equations followed at the time was research on the general form of the equations

$$ax^{2n} + x^n = c \qquad ax^{2n} + c = bx^n \qquad ax^{2n} = bx^n + c$$

as we can establish in the work of Abū Kāmil and Sinān ibn al-Fatḥ, amongst others.

Furthermore we witness, after al-Khwārizmī, the expansion of algebraic calculation. This, perhaps, is the principal theme of research, and the one more communally shared, amongst the algebraists following him. Thus even the terms of algebra have begun to be extended up to the sixth power of the unknown, as can be seen in the work of Abū Kāmil and Sinān ibn al-Fatḥ. Furthermore the latter[13] defines powers multiplicatively, in contrast with Abū Kāmil who gives an additive definition. But it is the algebraic work of Abū Kāmil which marks both this period and the history of algebra.[14] In addition to the expansion of algebraic calculation, he included into his book a new area of algebra, indeterminate analysis or rational Diophantine analysis. Thus, after having taken up the theory of equations with firmer demonstrations than those of his predecessor, he studies in a much more thorough and extensive manner the arithmetical operations on binomials and trinomials, demonstrating the result obtained each time. He states and justifies the sign rule and establishes calculation rules for fractions before passing to systems of linear equations with several unknowns and equations with irrational coefficients, such as

$$\left(x^2 + \frac{1}{\sqrt{2}}\,x\right)^2 = 4x^2 \qquad \frac{\sqrt{10x}}{(2+\sqrt{3})} = x - 10$$

Abū Kāmil integrates into his algebra the auxiliary numerical methods, of which some would have been contained in a lost book of al-Khwārizmī, such as

$$\sum_{k=1}^{n} a_k \qquad \sum_{k=1}^{n} k^2 \qquad \sum_{k=1}^{n} 2k$$

Abū Kāmil then studied numerous problems which lead to second degree equations.

We thus see that the research of al-Khwārizmī's successors, and especially Abū Kāmil, contributed to the theory of equations and to the extension of algebraic calculation to the field of rational numbers, and to the set of irrational numbers. The research of Abū Kāmil on indeterminate analysis had considerable repercussions in the development of this field, but it also gave him a new signification and a new status. Part of algebra, this analysis constitutes from now on a subject area in all treatments intended to cover the discipline.

THE ARITHMETIZATION OF ALGEBRA: AL-KARAJĪ AND HIS SUCCESSORS

We shall understand nothing of the history of algebra if we do not emphasize the contributions of two research movements which developed during the period previously considered. The first was directed towards the study of irrational quantities, whether as a result of a reading of book X of the *Elements* or in a way independently. We can recall, amongst many other mathematicians who took part in this research, the names of al-Māhānī, Sulaymān ibn ʿIṣma, al-Khāzin, al-Aḥwāzī, Yūḥannā ibn Yūsuf, al-Hāshimī and so on. It goes without saying that we cannot cover here these various contributions. We would just like to emphasize that, in the course of this work, calculation with irrational quantities was actively developed, and sometimes even parts of book X of the *Elements* of Euclid started being read in the light of the algebra of al-Khwārizmī. To take a single example, we consider that of al-Māhānī in the ninth century, who searched for the square root of five apotomes. Thus, to extract the square root of the first apotome,[15] al-Māhānī suggested that 'we proceed by the method of algebra and *al-muqābala*',[16] i.e. putting $a = x + y$ and $b = 4xy$, one obtains the equation $x^2 + b/4 = ax$. One then determines the positive root x_0, deduces y_0 and obtains

$$(a - \sqrt{b})^{1/2} = \sqrt{x_0} - \sqrt{y_0}$$

Al-Māhānī carries on thus for the next four apotomes, and for the second apotome $(\sqrt{b} - a)$ for example, with $b = 45$ and $a = 5$, he ends up with the equation

$$x^4 + \frac{625}{16} = \frac{65}{2}x^2$$

Now these mathematicians not only studied algebraic calculation of irrational quantities, but they were also to confirm the generality of algebra as a tool.

The second research movement was instigated by the translation of the

Arithmetica of Diophantus into Arabic, and notably by the algebraic reading of this last book. It is about 870 when Qusṭā ibn Lūqā translates seven books of Diophantus's *Arithmetica* under the significant title *The Art of Algebra.*[17] The translator used the language of al-Khwārizmī to reproduce the Greek of Diophantus, thus reorienting the contents of the book towards the new discipline. Now the *Arithmetica*, even if they were not a work on algebra in the sense of al-Khwārizmī, nevertheless contained techniques of algebraic calculation that were powerful for the time: substitution, elimination, changing of variables etc. They were the subject of commentaries by mathematicians such as Ibn Lūqā, their translator, in the ninth century, and Abū al-Wafā' al-Būzjānī a century later, but these texts are unfortunately lost. We know, however, that al-Būzjānī wanted to prove the Diophantine solutions in his commentary. This same Abū al-Wafā', in a text which is available to us, demonstrates the binomial formula, often used in the *Arithmetica*, for $n = 2.3.$[18]

Be that as it may, the progress of algebraic calculation, whether by its expansion into other areas or by the mass of technical results obtained, succeeded in rejuvenating the discipline itself. A century and a half after al-Khwārizmī, the Baghdad mathematician al-Karajī thought of another research project: the application of arithmetic to algebra, i.e. to study systematically the application of the laws of arithmetic and of certain of its algorithms to algebraic expressions and in particular to polynomials. It is precisely this calculation on the algebraic expressions of the form

$$f(x) = \sum_{k=-m}^{n} a_k x^k \qquad m, n \in Z_+$$

which has become the principal aim of algebra. The theory of algebraic equations is of course always present, but only occupies a modest place in the preoccupations of algebraists. One understands from that time on, that books on algebra undergo modifications not only in their content but also in their organization.

Al-Karajī devoted several writings to this new project, notably *al-Fakhrī* and *al-Badīʿ*. These books will be studied, reproduced and commented on by mathematicians until the seventeenth century, i.e. that the work of al-Karajī occupied the central place in research on arithmetical algebra for centuries, whilst the book of al-Khwārizmī became a historically important exposé, commented on only by second rate mathematicians. Without reproducing here the history of six centuries of algebra, we illustrate the impact of the work of al-Karajī by turning towards one of his successors in the twelfth century, al-Samaw'al (d. 1174). The latter includes in his algebra, *al-Bāhir*, the principal writings of al-Karajī and notably the two works cited previously. Al-Samaw'al begins by defining in a general way the notion of algebraic power[19] and, from the definition $x^0 = 1$, gives the rule equivalent to $x^m x^n = x^{m+n}$, $m, n \in Z$. Next comes the study of arithmetical operations on monomials and polynomials, notably those on the divisibility of polynomials, as well as the approximation of fractions by the elements of the ring of polynomials. We have for example

$$\frac{f(x)}{g(x)} = \frac{20x^2 + 30x}{6x^2 + 12} \approx \frac{10}{3} + \frac{5}{x} - \frac{20}{3x^2} - \frac{10}{x^3} + \frac{40}{3x^4} + \frac{20}{x^5} - \frac{80}{3x^6} - \frac{40}{x^7}$$

Al-Samaw'al obtains a sort of limited expansion of $f(x)/g(x)$ which is only valid for sufficiently large x.

We meet next the extraction of a square root of a polynomial with rational coefficients. But, in all the calculations on polynomials, al-Karajī had devoted a writing, lost now but luckily cited by al-Samaw'al, where he occupied himself with establishing the formula of binomial expansion and the table of its coefficients:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \qquad n \in N$$

It is during the demonstration of this formula that complete finite induction appears in an archaic form as the procedure of mathematical proof. Amongst the methods of auxiliary calculation, al-Samaw'al, following al-Karajī, gives the sum of different arithmetic progressions, with their demonstration:

$$\sum_{k=1}^{n} k, \ \sum_{k=1}^{n} k^2, \ \left(\sum_{k=1}^{n} k\right)^2, \ \sum_{k=1}^{n} k(k+1), \ \ldots$$

Next comes the response to the following question: 'How can multiplication, division, addition, subtraction and the extraction of roots be used for irrational quantities?'[20] The answer to this question led al-Karajī and his successors to reading algebraically, and in a deliberate manner, book X of the *Elements*, to extend to infinity the monomials and binomials given in this book and to propose rules of calculation, amongst which we find explicitly formulated those of al-Māhānī,

$$(x^{1/n})^{1/m} = (x^{1/m})^{1/n} \qquad \text{and} \qquad x^{1/m} = (x^n)^{1/mn}$$

with others such as

$$(x^{1/m} \pm y^{1/m}) = \{y[(x/y)^{1/m} \pm 1]^m\}^{1/m}$$

We also find an important chapter on rational Diophantine analysis, and another on the solution of systems of linear equations with several unknowns. Al-Samaw'al gives a system of 210 linear equations with ten unknowns.

From the works of al-Karajī, one sees the creation of a field of research in algebra, a tradition recognizable by content and the organization of each of the works. These, to reproduce the words of Ibn al-Bannā' in the thirteenth and fourteenth centuries, 'are almost innumerable'.[21] Citing here only a few, we find the masters of al-Samaw'al: al-Shahrazūrī, Ibn Abī Turāb, Ibn al-Khashshāb; al-Samaw'al himself, Ibn al-Khawwām, al-Tanūkhī, Kamāl al-Dīn al-Fārisī, Ibn al-Bannā' and, later, al-Kāshī, al-Yazdī etc.

In the midst of this tradition, the theory of algebraic equations, strictly speaking, is not central but nevertheless makes some progress. Al-Karajī himself considered quadratic equations, just like his predecessors. Certain of his successors, however, attempted to study the solution of cubic and quartic equations. Thus al-Sulamī, in the twelfth century, tackled cubic equations to find a solution by radicals.[22] The text of al-Sulamī testifies to the interest of the mathematicians of his time in a solution of cubic equations by radicals. He himself considers two types as possible:

$$x^3 + ax^2 + bx = c \qquad \text{and} \qquad x^3 + bx = ax^2 + c$$

However, he imposes the condition $a^2 = 3b$, and then gives for each equation a positive real root:

$$x = \left(\frac{a^3}{27} + c\right)^{1/3} - \frac{a}{3} \qquad \text{and} \qquad x = \left(c - \frac{a^3}{27}\right)^{1/3} + \frac{a}{3}$$

We can reconstruct the procedure of al-Sulamī as follows: by affine transformation he obtains the equation in its normal form; but, instead of finding the discriminant, he leaves aside the coefficient of the first power of the unknown to change the problem to one of extraction of a cubic root. Thus, for example, for the first equation, we take the affine transformation $x \rightarrow y - a/3$; the equation can be rewritten as

$$y^3 + py - q = 0$$

with

$$p = b - \frac{a^2}{3} \qquad \text{and} \qquad q = c + \frac{a^3}{27} + \left(b\,\frac{a}{3} - \frac{a^3}{9}\right)$$

Putting $b = a^2/3$, we have

$$y^3 = c + \frac{a^3}{27}$$

whence y, and therefore x.

Such attempts, attributed to the fourteenth-century Italian mathematician Master Dardi,[23] are frequent in the algebraic tradition of al-Karajī.

Thus, for example, the mathematician Ibn al-Bannā',[24] even though he recognizes implicitly the difficulty in solving cubic equations by radicals with the exception of $x^3 = a$ when he writes that, for equations which 'lead to other degrees (than the second), one cannot solve them by the method of algebra with the exception of "cubes equal to a number"', gives the equation

$$x^4 + 2x^3 = x + 30 \qquad (*)$$

which he solves in the following manner: one rewrites the equation

$$x^4 + 2x^3 + x^2 = x^2 + x + 30$$

which can be rewritten as

$$(x^2 + x)^2 = x^2 + x + 30$$

Putting $y = x^2 + x$, one has

$$y^2 = y + 30$$

On solving this equation, one has $y = 6$, and one can then solve $x^2 + x = 6$ to find $x = 2$, a solution of (*).

It is still too early to know exactly the contribution of mathematicians of this tradition to the solution of cubic and quartic equations; but this evidence, contrary to what we might think, shows that certain of them attempted to go much further than al-Karajī.

THE GEOMETRIZATION OF ALGEBRA: AL-KHAYYĀM

The algebraist arithmeticians kept to the solution of equations by radicals, and wanted to justify the algorithm of the solution. Sometimes even, from the same mathematician, Abū Kāmil for example, we come across two justifications, one geometric and the other algebraic. For cubic equations, he was missing not only their solution by radicals, but equally the justification of the solution algorithm, because the solution cannot be constructed with a ruler and a compass. The mathematicians of this tradition were perfectly aware of this fact, and one had written well before 1185: 'Since the unknown that one wants to determine and know in each of these polynomials is the side of the cube mentioned in each, and the analysis leads to the application of a known right-angled parallelepiped to a known line, and which is surplus to the entire parallelepiped by a cube or which is deficient by a cube; we can only do this synthese using conic sections'.[25] Now this recourse to conic sections, explicitly intended to solve cubic equations, quickly followed the first algebraic renderings of solid problems. We have mentioned in the ninth-century al-Māhānī and the lemma of Archimedes;[26] it was not long before other problems such as the trisection of an angle, the two means and the regular heptagon in particular were translated into algebraic terms. However, confronted with the difficulty mentioned above, and thus with that of solving cubic equations by radicals, mathematicians such as al-Khāzin, Ibn ʿIrāq, Abū al-Jūd ibn al-Layth, al-Shannī etc. ended up translating this equation into geometrical terms.[27] They then found in the course of studying this equation a technique already being used in the examination of solid problems, i.e. the intersection of conical curves. This is precisely the reason for the geometrization of the theory of algebraic equations. This time, in contrast with Thābit ibn Qurra, one does not look for a geometrical translation of algebraic equations to find the geometrical equivalent of the algebraic solution already obtained, but to determine, with the help of geometry, the positive roots of the equation that have not yet been found by other means. The attempts of al-Khāzin, al-Qūhī, Ibn al-Layth, al-Shannī, al-Bīrūnī etc. are just partial contributions until the conception of the project by al-Khayyām: the elaboration of a geometrical theory for equations of degree equal to or less than 3. Al-Khayyām (1048–1131) intended first to supersede the fragmentary research, i.e. the research linked in one or another form with cubic equations, in order to elaborate a theory of equations and to propose at the same time a new style of mathematical writing. Thus he studied all types of third degree equations, classed in a formal way according to the distribution of constant terms, of first degree, of second degree and of third degree, between the two members of the equation. For each of these types, al-Khayyām found a construction of a positive root by the intersection of two conics. Thus for example to solve the equation 'a cube is equal to the sides plus a number', i.e.

$$x^3 = bx + c \qquad b, c > 0 \tag{*}$$

al-Khayyām considered only the positive root. To determine it, he proceeded from the intersection of a semi-parabola

$$P = \{(x, y); b^{1/2}y = x^2\}$$

and a branch of an equilateral hyperbola having the same vertex:

$$H = \left\{(x, y); y^2 = \left(\frac{c}{b} + x\right)x\right\}$$

He showed that they have a second common point which corresponds to the positive root. We note that, if one takes the parabola and the hyperbola, for certain values of b and c, the points of intersection which correspond to the negative roots, can be obtained (Figure 11.2).

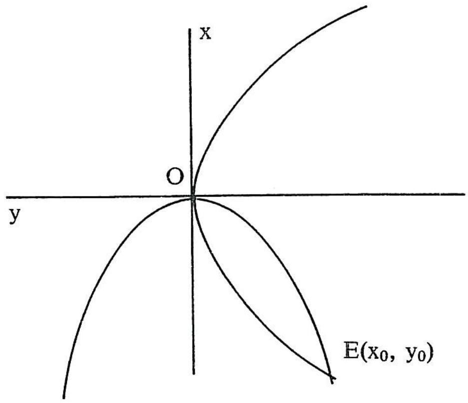

Figure 11.2

Thus, for the choice of curves, we note that if we introduce a trivial solution $x=0$ then equation (*) becomes

$$\frac{x^4}{b}=x^2+\frac{c}{b}x$$

whence we obtain the two preceding curves. From their intersection (x_0, y_0)

$$\frac{b^{1/2}}{x_0}=\frac{x_0}{y_0}=\frac{y_0}{x_0+c/b}$$

whence

$$\frac{b}{x_0^2}=\frac{x_0}{x_0+c/b}$$

and x_0 is the solution of equation (*).

To work out this new theory, al-Khayyām is forced to conceive and formulate new and better relations between geometry and algebra. We recall in this regard that the fundamental concept introduced by al-Khayyām is the unit of measure which, suitably defined with respect to that of dimension, allowed the application of geometry to algebra. Now this application led al-Khayyām in two directions, which could seem at first to be paradoxical: whilst algebra was now identified with the theory of algebraic equations, this seemed from now on, but still hesitantly, to transcend the split between algebra and geometry. The theory of equations is more than ever a place where algebra and geometry meet and, more and more, analytical arguments and methods. The real evidence of this situation is the appearance of memoirs dedicated to the theory of equations, such as that of al-Khayyām. Contrary to algebraist arithmeticians, al-Khayyām moves away from his treatise the chapters on polynomials, on polynomial arithmetic, on the study of algebraic irrationals etc. He also creates a new style of mathematical writing: he begins with a discussion of the concept of algebraic magnitude, to define the concept of measure unit; he advances next the necessary lemmas as well as a formal classification of equations – according to the number of terms – before then examining, in order of increasing difficulty, binomial equations of second degree, binomial equations of third degree, trinomial equations of second degree, trinomial equations of third degree, and finally equations containing the inverse of the unknown. In his treatise, al-Khayyām reaches two remarkable results that historians have usually attributed to Descartes: a general solution of all equations of third degree by the intersection of two conics; and a geometrical calculation made possible by the choice of a unit length, keeping faithful, in contrast to Descartes, to the homogeneity rule.

Al-Khayyām, we note, does not stop there but tries to give an approximate numerical solution for cubic equations. Thus, in a memoir entitled *On the division of a quarter of a circle*,[28] in which he announces his new project on the theory of equations, he reaches an approximate numerical solution by means of trigonometrical tables.

THE TRANSFORMATION OF THE THEORY OF ALGEBRAIC EQUATIONS: SHARAF AL-DĪN AL-ṬŪSĪ

Until recently, it was thought that the contribution of mathematicians of this time to the theory of algebraic equations was limited to al-Khayyām and his work. Nothing of the kind. Not only did the work of al-Khayyām begin a real tradition but, in addition, it was profoundly transformed barely a half century after his death.

According to historical evidence the student of al-Khayyām, Sharaf al-Dīn al-Masʿūdī,[29] would have written a book which treated the theory of equations and the solution of cubic equations. But this book, if it was written, has not reached us at all. Two generations after al-Khayyām, we encounter one of the most important works of this movement: the treatise of Sharaf al-Dīn al-Ṭūsī *On the Equations.*[30] Now this treatise of al-Ṭūsī (about 1170) makes some very important innovations with respect to that of al-Khayyām. Unlike the approach of his predecessor, that of al-Ṭūsī is no longer global and algebraic but local and analytic. This radical change, particularly important in the history of classical mathematics, necessitates that we consider it a little longer.

The *Treatise* of al-Ṭūsī opens with the study of two conical curves, used in the following. It concerns a parabola and a hyperbola, to which is added a circle assumed known, to exhaust all the curves to which the author had recourse. He seems to suppose that his reader is familiar with the equation of a circle, obtained from the power of a point with respect to it, and uses this preliminary part to establish the equation of a parabola and the equation of an equilateral hyperbola, with respect to two systems of axes.

Next follows a classification of equations of degree less than or equal to 3. In contrast with al-Khayyām, he opts for an extrinsic criterion of classification rather than intrinsic. Whilst al-Khayyām, as we have noted, organizes his exposition according to the number of monomials which form the equation, al-Ṭūsī chooses as the criterion the existence or not of positive solutions; i.e. the equations are arranged according to whether they allow 'impossible cases' or not. One easily understands then that the *Treatise* is made up of only two parts, corresponding to the preceding alternatives. In the first part, al-Ṭūsī deals with the solution of twenty equations; for each case, he proceeds through a geometrical construction of roots, the determination of the discriminant for the only quadratic equations, and finally to the numerical solution with the help of the method known as Ruffini–Horner. He reserves the application of this method to polynomial equations, and not just to the extraction of the root of a number.

Already, we can therefore spot the constituent elements of the theory of equations of the twelfth century, in the tradition of al-Khayyām: geometrical construction of roots, numerical solution of equations, and finally recall of the solutions by radicals of the quadratic equation, this time rediscovered from geometrical construction. In the first part, after having studied second degree equations and the equation $x^3 = c$, al-Ṭūsī examines eight third degree equations. The first seven all have a single positive root. They can have negative roots that al-Ṭūsī did not recognize. To study each of these equations, he chooses two second degree curves or, more precisely, two curved segments. He shows, through geometric means, that the arcs under study have a point of intersection whose abscissa verifies the proposed equation (they can have other points of intersection). The geometrical properties described by al-Ṭūsī are, aside from some details which he passes over though they are satisfied by the data he chooses, the characteristic properties and thus lead to the equations of the curves under consideration. Thanks to the use of the terms 'interior' and 'exterior', al-Ṭūsī can employ the continuity of the curves and their convexity. We can thus translate his approach to the equation

$$x^3 - bx = c \qquad b, c > 0$$

He considered in fact the two expressions

$$g(x)=\left[x\left(\frac{c}{b}+x\right)\right]^{1/2} \quad \text{and} \quad f(x)=\frac{x^2}{\sqrt{b}}$$

and showed that, if α and β exist such that $(f-g)(\alpha)>0$ and $(f-g)(\beta)<0$, then there exists $\gamma \in]\alpha, \beta[$ such that $(f-g)(\gamma)=0$.

In the reading of this first part, we see that, as with al-Khayyām, al-Ṭūsī studies principally the geometric construction of positive roots of the twenty equations of degree less than or equal to 3, since those which are left are transformed by means of affine transformations to one or other of these types. In an analogous method to that of al-Khayyām, he begins with plane geometrical constructions if the equation, reduced as much as possible, is of first or second degree and by constructions using two or three of the curves mentioned if the equation, reduced as much as possible, is cubic.

Although the first part of the *Treatise* is closely dependent on the contribution of al-Khayyām, one already perceives some differences, of which the consequences only appear in the second part. For each equation studied, al-Ṭūsī demonstrates the existence of a point of intersection of two curves, while al-Khayyām only really undertook this study for the twentieth equation. Al-Ṭūsī has also introduced some ideas which he will have recourse to frequently in the second part, such as affine transformations and the distance of a point from a line.

The second part of the *Treatise* is dedicated to five equations which, according to the expression of al-Ṭūsī, allow 'impossible cases', i.e. cases where there is no positive solution. They are the equations

(1) $x^3+c=ax^2$
(2) $x^3+c=bx$
(3) $x^3+ax^2+c=bx$
(4) $x^3+bx+c=ax^2$
(5) $x^3+c=ax^2+bx$

In contrast with al-Khayyām, al-Ṭūsī could not be content with a simple statement of these 'impossible cases'. Preoccupied with the proof of the existence of points of intersection, and consequently with the existence of roots, he had to characterize such cases and look for their justification. Now it is precisely the meeting of this technical problem and the questioning which followed which brought al-Ṭūsī to break from the tradition of al-Khayyām and to modify his initial project. But, to comprehend this major change, it is necessary to analyse the approach of al-Ṭūsī.

Each of the five equations are written in the form $f(x)=c$; f is a polynomial. To characterize the 'impossible cases', al-Ṭūsī studies in fact the intersection of the curve $y=f(x)$ with the line $y=c$. For al-Ṭūsī, it is a segment of the curve, that for which we have simultaneously $x > 0$ and $y = f(x) > 0$, a segment that may not exist. We note that, for him, the problem only makes sense if $x > 0$ and $f(x) > 0$, and in each case he poses the condition so that $f(x)$ is strictly positive. Thus, in equation (1) he poses the condition $0 < x < a$, in equation (2) $0 < x < \sqrt{b}$; in (3) he gives the condition $0 < x < \sqrt{b}$, which is, however, not sufficient. Al-Ṭūsī is therefore constrained to examine the relationship between the existence of solutions and the position of the constant c with respect to the maximum of the polynomial function. It is on this occasion that he introduces new concepts, new procedures and a new language; and in addition, he defines a new subject. He thus begins by formulating the concept of the maximum of an algebraic expression, which he calls 'the largest number' – *al-ʿadad al-aʿẓam*. Let $f(x_0) = c_0$ be the maximum; this gives the point (x_0, c_0). Al-Ṭūsī determines next the roots of $f(x) = 0$, i.e. the intersection of the curve with the abscissa; finally he deduces a double inequality for the roots of $f(x) = c$.

The whole problem from now on is therefore for him to find the value of x which yields a maximum of $f(x)$. Al-Ṭūsī proceeds then by solving an equation which turns out to be, though in a different notation, $f'(x) = 0$, where f' is the polynomial, derivative of f. But, before examining this central problem of the derivative, we note the change and the introduction of local analysis. We begin by recalling the results of al-Ṭūsī. For equation (1) the derivative admits two roots, 0 and $2a/3$, which give respectively a minimum $f(0) = 0$ and a maximum $f(2a/3) = c_0$. On the other hand, the equation $f(x) = 0$ admits a double root $\lambda_1 = 0$ and a positive root $\lambda_2 = a$. Al-Ṭūsī therefore concludes: if $c < c_0$ equation (1) has two positive roots x_1 and x_2 such that $\lambda_1 = 0 < x_1 < x_0 < x_2 < \lambda_2 = a$. Notice that a third, negative, root x_3 exists, which al-Ṭūsī did not consider. For equations (2), (3) and (5) his reasoning is analogous. In these three cases, the derivative admits two roots with opposite signs. The positive root x_0 gives the maximum $c_0 = f(x_0)$ and the equation $f(x) = 0$ admits three simple roots, of which one is negative and the others are $\lambda_1 = 0$ and λ_2 – whence the conclusion obtained previously. To best illustrate the approach of al-Ṭūsī, we resume his discussion of equation (1). This equation can be rewritten

$$c = x^2(a - x) = f(x)$$

Al-Ṭūsī considered three cases.

$$c > \frac{4a^3}{27}$$

The problem is impossible, according to al-Ṭūsī (it admits a negative root).

$$c = \frac{4a^3}{27}$$

Al-Ṭūsī determines the double root $x_0 = 2a/3$ (but does not recognize the negative root).

$$c < \frac{4a^3}{27}$$

Al-Ṭūsī determines two positive roots, with

$$0 < x_1 < \frac{2a}{3} < x_2 < a$$

He then studies the maximum of $f(x)$; he shows that

$$f(x_0) = \sup_{0<x<a} f(x) \qquad \text{with } x_0 = \frac{2a}{3} \qquad (*)$$

by first showing that

(a) $$x_1 > x_0 \Rightarrow f(x_1) < f(x_0)$$

followed by

(b) $$x_2 < x_0 \Rightarrow f(x_2) < f(x_0)$$

and from (a) and (b) he gets (*).

To find $x_0 = 2a/3$, al-Ṭūsī solves $f'(x) = 0$. He next calculates

$$f(x_0) = f\left(\frac{2a}{3}\right) = \frac{4a^3}{27}$$

which allows him to justify the three cases considered previously. He next determines the two positive roots x_1 and x_2. He puts $x_2 = x_0 + y$; this affine transformation leads to the equation

$$y^3 + ay^2 = k$$

with $k = c_0 - c = 4a^3/27 - c$, an equation already solved by al-Ṭūsī in the first part of the *Treatise*. He next justifies this affine transformation. He uses also the affine transformation $x_1 = y + a - x_2$, with y a positive solution of an equation solved earlier in the *Treatise*. Al-Ṭūsī justifies this last affine transformation and finally shows that $x_1 \neq x_0$ and $x_1 \neq x_2$.

In equation (4) there suddenly appeared a difficulty, since the maximum $f(x_0)$ could be negative. Al-Ṭūsī then imposes a necessary condition, to consider only the case where $f(x_0) > 0$, and next proceeds as before. The equation $f'(x) = 0$ has then two roots x_0' and x_0 ($x_0' < x_0$), which correspond respectively to a negative minimum and a positive maximum. Al-Ṭūsī only considers the root x_0 and obtains $c_0 = f(x_0)$. However, the equation $f(x) = 0$ has, in this case, three roots, 0, $\lambda_1 > 0$, $\lambda_2 > 0$, with $\lambda_1 < \lambda_2$. Al-Ṭūsī deduces that, for $c < c_0$, equation (4) has two positive roots x_1 and x_2 such that

$$0 < \lambda_1 < x_1 < x_0 < x_2 < \lambda_2$$

This quick summary shows that the presence of the idea of the derivative is neither fortuitous nor secondary but, rather, intentional. It is true, however, that this is not the first time that one encounters the expression of a derivative in the *Treatise*: it is already introduced by al-Ṭūsī to construct a numerical method of solution of equations. This method goes as follows: al-Ṭūsī determines the first decimal digit of the root, as well as its decimal order. The root is then written as $x = s_0 + y$, with $s_0 = \sigma_0 \times 10^r$ (r the decimal order). He determines next the second digit with the help of the equation in y, $f(s_0 + y) = 0$; this algorithm, called Ruffini–Horner, is used to determine the different terms of the preceding cubic equation in y. The algorithm introduced by al-Ṭūsī serves to arrange the calculations so as to minimize the number of necessary multiplications, and is none other than a slightly modified form of the Ruffini–Horner algorithm adapted for cubic equations. Al-Ṭūsī then introduces as the coefficient of y the value $f'(s_0)$ of the derivative of f at s_0. Al-Ṭūsī obtains the last digit of y, i.e. the second digit of the required root, by taking the integer part of

$$-f(s_0)/f'(s_0)$$

We recognize here the method known as 'Newton's' for the approximate solution of equations. After having determined the second digit, which is the first of y, one applies the same algorithm to the equation in y to find a third digit, and one continues like this until the root is obtained, which is an integer in the cases considered by al-Ṭūsī.[31] However, if it were not, one finds the numbers after the decimal point by continuing as before. The successors of al-Ṭūsī proceeded in this way for the case where the root is not an integer, as explained in the text of al-Aṣfahānī, in the nineteenth century.[32]

If the presence of an expression for the derivative is not doubted, it remains that al-Ṭūsī did not explain the route which led him to such a notion. To understand better the originality of his method, we consider the example of equation (3) which is written

$$f(x) = x(b - ax - x^2) = c$$

The fundamental problem is to find the value $x = x_0$ at which the maximum is reached. Now it is in explaining the splitting of equation (3) into two equations solved beforehand by means of affine transformations

$$x \to y = x - x_0 \qquad \text{and} \qquad x \to y = x_0 - x$$

that al-Ṭūsī gives

$$f(x_0) - f(x_0 + y) = 2x_0(x_0 + a)y - (b - x_0^2)y + (3x_0 + a)y^2 + y^3$$

and

$$f(x_0) - f(x_0 - y) = (b - x_0^2)y - 2x_0(x_0 + a)y + (3x_0 + a)y^2 - y^3$$

Al-Ṭūsī has to compare $f(x_0)$ with $f(x_0 + y)$ and $f(x_0 - y)$ noting that on $]0, \lambda_2[$ the terms

$$y^2(3x_0 + a + y) \qquad \text{and} \qquad y^2(3x_0 + a - y)$$

are positive. Next, he can deduce two equalities such that

$$\begin{array}{ll} \text{if } b - x_0^2 \geqslant 2x_0(x_0 + a) & \text{then } f(x_0) > f(x_0 + y) \\ \text{if } 2x_0(x_0 + a) \geqslant b - x_0^2 & \text{then } f(x_0) > f(x_0 - y) \end{array}$$

and in consequence

$$b - x_0^2 = 2x_0(x_0 + a) \Rightarrow \begin{cases} f(x_0) > f(x_0 + y) \\ f(x_0) > f(x_0 - y) \end{cases}$$

i.e. if x_0 is the positive root of the equation

$$f'(x) = b - 2ax - 3x^2 = 0$$

then $f(x_0)$ is the maximum of $f(x)$ in the interval studied. We notice that the two equalities correspond to the Taylor expansion with

$$f'(x_0) = b - 2ax_0 - 3x_0^2 \qquad \frac{1}{2!} f''(x_0) = -(3x_0 + a)$$

$$\frac{1}{3!} f'''(x_0) = -1$$

This method of al-Ṭūsī consists then, it seems, of arranging $f(x_0 + y)$ and $f(x_0 - y)$ according to powers of y, and of showing that there is a maximum when the coefficient of y is zero in this expansion. The value of x for which $f(x)$ is maximum is therefore the positive root of the equation represented by $f'(x) = 0$. The virtue of the affine transformations $x \to x_0 \pm y$, with x_0 the root of $f'(x) = 0$, is that the term in y in the new equation vanishes. It is probable that starting from this property that al-Ṭūsī discovered the derivative equation $f'(x) = 0$, perhaps together with consideration of the graph representing f which he never draws in the *Treatise*. For small y, the principal part of the variation of $f(x_0 \pm y)$ is in y^2 and does not change sign with y. I have shown elsewhere that the method of al-Ṭūsī resembles strongly that of Fermat, in the latter's investigation of maxima and minima of polynomials.[33]

As we have just seen, the theory of equations is no longer only an area of algebra but covers a much wider domain. The mathematician gathers within this theory the geometrical study of equations and their numerical solution. He poses and solves the problem of the possible conditions for each equation, which leads him to devise the local study of curves that he uses, and notably to study systematically the maximum of a third degree polynomial by means of the derivative equation. In the course of the numerical solution, he does not only apply certain algorithms where one meets again the idea of the derivative of a polynomial, but he tries hard to justify these algorithms with the help of the idea of 'dominant polynomials'. It is clear that it is a mathematics of a very high level for this epoch; put simply, here already we touch the limits of a mathematical research carried out without efficient symbolism. All the research of al-Ṭūsī was done in fact in natural language, without any symbolism (except, perhaps, for a certain symbolic use in tables), which made it particularly complicated. One such difficulty appears as an obstacle, not only to the internal progress of the research itself, but also to the communication of results. In other words, as soon as a mathematician handles analytical ideas, such as those mentioned above, natural language is found to be inadequate to express the concepts and operations which were applied, constituting a limit to innovation in, as well as to diffusion of, this mathematical knowledge. The followers of al-Ṭūsī were, in all likelihood, affected by this same obstacle, until mathematical notation was truly transformed, after Descartes especially.

But the example of al-Ṭūsī suffices to show that the theory of equations not only was transformed after al-Khayyām but did not cease to demarcate itself even more clearly from search for solutions by radicals; it thus ended up by covering a vast field, including sectors which later would belong to analytical geometry, or even to analysis.

But what was the destiny of this theory of equations of al-Ṭūsī? This question is still at the edges of research, and we cannot at this time provide a satisfactory answer. We do not know of work in algebra by his student Kamāl al-Dīn ibn Yūnus. However, the student Athīr al-Dīn al-Abharī (d. 1262) of Kamāl al-Dīn ibn Yūnus composed an algebra which reached us abridged, according to the copyist. But, in the part that we do have, he applies the method of numerical solution of al-Ṭūsī, and in the same terms, to the equation $x^3 = a$. Al-Khilāṭī,[34] another algebraist of this time, recalls that al-Ṭūsī was 'the master of his master' and that he studied cubic equations, but he himself was faithful to the tradition of al-Karajī. Other evidence of this time mentions al-Ṭūsī,[35] but nothing has come down to us which indicates that one or the other of the mathematicians had taken up the theory of al-Ṭūsī. Whilst in effect we find traces of the book of al-Ṭūsī with his followers, we do not know for the moment of any commentaries on his algebra. Such could have existed but, even if this were the case, we doubt that it could surpass the work of al-Ṭūsī without setting out the operative notation necessary to develop the analytical ideas already contained in al-Ṭūsī's *Treatise on Equations*.

NOTES

1 In the preamble of his book, al-Khwārizmī mentions the generous encouragement of the arts and sciences by the Caliph al-Ma'mūn, who had encouraged him to write his book. Now al-Ma'mūn reigned between 813 and 833, which consequently are the limits for the dating of the book. Cf. al-Khwārizmī, *Kitāb al-jabr*.

2 The title of the book is *Kitāb al-jabr wa al-muqābala*. Recall that the two terms *al-jabr* and *al-muqābala* refer to both a discipline and two operations. Consider for example

$$x^2 + c - bx = d \qquad \text{with } c > d$$

Al-jabr consists in transposing the subtractive expressions

$$x^2 + c = bx + d$$

and *al-muqābala* in reducing to similar terms:

$$x^2 + (c - d) = bx$$

3 Cf. Al-Khwārizmī, *Kitāb al-jabr*, p. 16.

4 Thus, Abū Kāmil writes about al-Khwārizmī: 'The one who first achieved a book of algebra and of *al-muqābala*; the one which started and invented all the fundamentals found there'; Abū Kāmil, MS Kara Mustafa, 379, folio 2^r. The same Abū Kāmil wrote: 'I have established, in my second book (*al-Waṣāya bi-al-jabr*) the proof of authority and priority in algebra and *al-muqābala* of Muḥammad ibn Mūsā al-Khwārizmī, and I replied to a hot-head called Ibn Barza, about what he attributed to ʿAbd al-Ḥāmid, whom he mentioned was his grandfather.' Cf. Ḥajjī Khalīfa, vol. 2, pp. 1407–8. We can multiply the evidence which is abundant in this area. Sinān ibn al-Fatḥ, in the introduction to his pamphlet, only mentions al-Khwārizmī, confirming that algebra was his creation: 'Muḥammad ibn Mūsā al-Khwārizmī wrote a book which he called algebra and *al-muqābala*'.

5 Cf. Al-Khwārizmī, *Kitāb al-jabr*, pp. 20–1.

6 *Ibid.*, pp. 21–2.

7 *Ibid.*, p. 27.

8 Cf. Aydin Sayili, pp. 145 *et seq.*

9 Thābit ibn Qurra: *Fī taṣḥīḥ masā'il al-jabr bi-al-barāhīn al-handasiyya*, MS Topkapi Saray, Ahmet III, no. 2041, folio 245^r.

10 *Ibid.*, folio 246^v.

11 It is an anonymous manuscript (no. 5325 Astan Quds, Meshhed, folio 24^{r-v}) falsely attributed to Abū Kāmil; copied in 581 H/1185.

12 See later, note 24.

13 On powers in Sinān ibn al-Fatḥ, see Rashed (1984: 21 n.11).

14 Abū Kāmil, see note 4.

15 Let $a+\sqrt{b}$ be first binomial, i.e.

$$a \in Q \qquad b \in Q \qquad a > \sqrt{b} \qquad \sqrt{b} \notin Q \qquad \frac{(a^2-b)^{1/2}}{a} \in Q$$

Then $a-\sqrt{b}$ is a first apotome.

16 Al-Māhānī, *Tafsīr al-maqāla al-ʿāshira min kitāb Uqlīdis*, MS BN Paris 2457, folios 180^{v}–187^{r} (cf. especially folio 182^{r}).

17 Cf. Diophantus, *Les Arithmétiques*.

18 Abū al-Wafā' al-Būzjānī: *Fī jamʿ aḍlāʿ al-murabbaʿāt wa al-mukaʿʿabāt wa akhdh tafāḍulahā*, MS 5521 Astan Quds, Meshhed.

19 This is what al-Samaw'al writes, after having noted in a table, on either side of x^0, the powers: 'If the two powers are on either side of unity, from one of them we count in the direction of unity the number of elements in the table which separate the other power from unity, and the number is on the side of unity. If the two powers are on the same side of unity, we count in the opposite direction to unity' (al-Samaw'al, French Introduction, p. 19).

20 *Ibid.*, p. 37.

21 Ibn al-Bannā': *Kitāb fī al-jabr wa al-muqābala*, MS Dar al-Kutub, Riyāḍa, M., folio 1.

22 Al-Sulamī: *Al-muqaddima al-kāfiya fī ḥisāb al-jabr wa al-muqābala*, Collection Paul Sbath, no. 5, folios 92^{v}–93^{r}.

23 Cf. van Egmond, (1983).

24 Ibn al-Bannā', *Kitāb fī al-Jabr*, folio 26^{v}.

25 Compare the manuscript falsely attributed to Abū Kāmil, n.11, folio 25.

26 Here is how al-Khayyām recounts this history in his own way in his celebrated treatise on algebra: 'As for the Ancients, nothing has come down to us of what they said: perhaps, after having researched and examined them, they had not grasped them; perhaps their research had not obliged them to examine; perhaps finally nothing of which they said has been translated into our language. As for the Moderns, it is al-Māhānī [Abū ʿAbdallāh Muḥammad b. ʿĪsā Aḥmad al-Māhānī, he lived between about 825 and 888] who amongst them has led to the algebraic analysis of the lemma that Archimedes used, considering it as admitted, in proposition 4 of the second book of his work on *The Sphere and the Cylinder*; now he has arrived at cubes, squares and numbers forming an equation that he does not succeed in solving even after much thought; he thus ended by judging that it was impossible, until Abū Jaʿfar al-Khāzin appeared and solved the equation by conic sections.

Following him, some geometricians had need of several types of [these equations], and certain of them solved some; but none of them said anything definite about the enumeration of their types, nor about [the means] of obtaining the forms of each of them, nor about their demonstrations, except for the two types that I will mention.

As for me, I have wished, and still ardently do, to know with certitude all these types, and to distinguish, amongst the forms of each of them, the possible cases from the impossible cases, through demonstrations; I know in effect that one has a very urgent need when one is wrestling with the difficulties of the problems. However, I have not been able to dedicate myself exclusively to the acquisition of this, nor to think about it with perseverance, distracted as I have been by vicissitudes. For we find ourselves tried by the dwindling of the men of science, with the exception of a group as small as its afflictions are large, and whose worry is to find time on the wing to dedicate to the achievement and the perfectioning of the science'. This text is fundamental for the history of cubic equations. See our edited and translated version, with commentary, of *L'Oeuvre Algébrique d'al-Khayyām*, pp. 11–12.

27 *Ibid.*, pp. 82–4: 'As for the ancient mathematicians, who did not speak in our language, they had attracted attention to nothing of all of this, or nothing has reached us which has been translated into our language.

And among the Moderns, who speak our language, the first who had need of a trinomial sort of these fourteen kinds is al-Māhānī, the geometrician. He solved the lemma which Archimedes has taken, considering it as admitted, in proposition 4 of the second book of his work on *The Sphere and the Cylinder*. It is this which I am going to explain.

Archimedes said: the two straight lines AB and BC are of known magnitude, and one is in the prolongation of the other; and the ratio of BC to CE is known. CE is therefore known, as is shown in the *Data* [of Euclid]. He then said: let us set the ratio of CD to CE equal to the ratio of the square of AB to the square of AD.

He did not say how this was known, since one had necessarily to have conic sections. And, besides this, he introduced nothing in the book which was founded on conic sections. He also took this as admitted. The fourth proposition concerns the division of a sphere by a plane, according to a given ratio. But al-Māhānī used the terms of algebraists in order to facilitate [the construction]; as the analysis led to numbers, squares and cubes in equations, and as he could not solve them by conic sections, he thus ended by saying that it is impossible. The solution of one of these types therefore remained hidden from this eminent man, in spite of his eminence and his primacy in this art, until Abū Ja'far al-Khāzin appeared and indicated a method which he described in his treatise; and Abū Naṣr b. 'Irāq, protégé of the Prince of Believers from the land of Khwārizm, solved the lemma that Archimedes had assumed to determine the side of a heptagon inscribed in a circle, and which is founded on the square verifying the mentioned property: he used algebraic terms. The analysis led to [the equation] "a cube plus squares equal a number", which he solved by sections.

This man, by my life, is of an excellent class in mathematics. This is the problem in the face of which Abū Sahl al-Qūhī, Abū al-Wafā' al-Būzjānī, Abū Ḥāmid al-Ṣāghānī, and a group of their colleagues, who were all devoted to His Lordship 'Aḍud al-Dawla, in the City of Peace [Baghdad], were found to be powerless; the problem, I say, is as follows: if you divide ten into two parts, the sum of their squares plus the quotient of the largest over the smallest is seventy-two. Analysis leads to squares equal to the roots plus a number. These eminent men were totally perplexed for a long time when faced with this problem, until Abū al-Jūd solved it. They have conserved [his solution] in the library of the Samanid kings. There are thus three kinds of compound equations, two trinomials and a quadrinomial. The only binomial equation, i.e. "the cube is equal to a number", our eminent predecessors solved. Nothing from them has reached us about the ten [equations] which remain, nor anything as detailed. If time granted us and if success accompanies me, I will record these fourteen types with all their branches and sections, distinguishing between the possible and impossible cases – in fact certain of these types require some conditions for them to be valid – in a treatise which will contain many of the lemmas preceding them, of great utility for the principles of this art.'

28 *ibid.*, p. 80.

29 See Rashed (1974b).

30 Cf. Sharaf al-Dīn al-Ṭūsī.

31 Taking the example of the numerical resolution of the equation

$$x^3 = bx + N$$

al-Ṭūsī writes:

> To determine the required number, we place the number in the table and we count its rows by cubic root, no cubic root, cubic root. We place the zeros of the cubic root, we count also [the rows] of the number by root, no root, until we arrive at the homonymous root of the last place assigned a cubic root. We next place the number of roots, and we count the rows by root, no root. The homonymous row of the last place assigned a root for this number of roots is the last row of the root of the number of roots. The problem has two cases.
>
> *First case*: The homonymous root in the last place assigned a cubic root is much greater than [the row] of the last part of the number of roots, as when we say: a number of the form 3 2 7 6 7 0 3 8 plus nine hundred and sixty-three roots equals a cube. We count from the homonymous root of the last place assigned a cubic root until the last row of the number of roots, and we count the same number from the last place assigned a cubic root in this direction; and there where we end up, we place the last part of the number of roots reduced to one third; we then have this figure:
>
> 3 2 7 6 7 0 3 8
> 3 2 1
>
> Since the homonymous root of the last place assigned a cubic root is the third place assigned a root, it is in the row of tens of thousands which is higher than the last row of the number of roots, which is in [the row] of hundreds. We count from the row of the homonymous root of the last place assigned a cubic root until the hundreds, and we count through this number also from the row of the last place assigned a cubic root, finishing in the tens of thousands; we place the last part of the third of the number of roots in this row and we place next the required cubic root, which is three, at the place of the last zero. We subtract its cube from what is beneath it, we multiply it by the rows of a third of the number of roots and we add three times the product to the number. We put the square of the required

number parallel to itself under the number, according to this figure:

3
6 0 5 5 9 3 8
3 2 1
9

We subtract the third of the number of roots from the square of the required number and we remove the third of the number of roots; there remains then this figure:

3
6 0 5 5 9 3 8
8 9 6 7 9

We move the upper line by two rows and the lower line by one row; we place the second required number, two, and we subtract its cube from the number; we multiply it by the first required number, we add the product to the lower line, we multiply it by the lower line and we subtract three times each product of the number; we add the square of the second required number to the lower line, we multiply it by the first required number, we add the product to the lower line and we move the upper line by two rows and the lower line by one row. We place another required number, which is one; we subtract its cube from the number, we multiply it by the first required number and the second, we add the result to the lower line, we multiply it by the lower line and we subtract three times this product from the number. The upper line is then the figure 3 2 1 which is the required root.

Second case: The last row of the number of roots is greater than the homonymous root of the last place assigned a cubic root, as when we say: a number of roots equal to 1 02021 plus a number of the form 3 2 7 4 2 0 equal a cube. We count the number of roots by root, no root, and we add to the number two rows by putting zeros in front of it; we look for the place assigned the highest root corresponding to the number of roots; we then place the zeros of the cubic root and then we look for the highest homonymous cubic root of this place assigned a root. We move the row of the number of roots parallel to this root, so that it is parallel to the cubic root which is homonymous. We place the other rows of the number of roots in order; one has then this figure:

0 0 3 2 7 4 2 0
1 0 2 0 2 1

because the highest place assigned a root which corresponds to them is the third, and it is in the column of tens of thousands: its homonymous is the third place assigned a cubic root which is in the [column] of thousands of thousands. We move the row of tens of thousands of the number of roots so that it is parallel to the place assigned the third cubic root, and we look for the greatest number such that one can remove its square of the number of roots; it is three; we place it in the third place assigned a cubic root; we multiply it by the rows of the number of roots, we add the product to the number and we remove its cube of the number. We reduce the number of roots to a third; it will begin then at the row of hundreds according to this figure:

3
3 9 3 3 7 2 0
3 4 0 0 7

We place next the square of the required number parallel to it below the number; one subtracts the third of the number of roots from it and deletes the line which is the third of the number of roots; we move the upper line by two rows and the lower line by one row and we apply the procedure until it is finished.

(Sharaf al-Dīn al-Ṭūsī, vol. I, pp. 49–52; see Table VI, p. cvii and Table VII, p. cviii)

32 Sharaf al-Dīn al-Ṭūsī, vol. I, pp. 118 *et seq.* On the other hand, al-Aṣfahānī gives in the same treatise an interesting method for finding a positive root of a cubic equation, based on the property of a fixed point. Did he take it from his ancient predecessors, as he did for the method of al-Ṭūsī? He probably did, but at this time we cannot settle such a question. Here, described quickly, the method is applied to the same example of al-Aṣfahānī. Solve the equation

$$x^3 + 210 = 121x \qquad \text{with} \qquad x \in R_+$$

We write this equation in the form

$$x = (121x - 210)^{1/3} = f(x)$$

Al-Aṣfahānī takes then $x_1' = 11$, whence

$$y_1 = f(x_1') = (1121)^{1/3} < 11$$

He takes an approximate value by default of y_1, namely 10.3; he finds

$$f(10.3) = (1036.3)^{1/3} < 10.3$$

He takes then $x_2' = 10.3$ and $y_2 = f(x_2') = (1036.3)^{1/3}$. He takes next an approximate value by default of y_2, namely 10.1. He finds that

$$f(10.1) = (1012.1)^{1/3} < 10.1$$

He takes then $x_3' = 10.1$ and so on; the first terms of this series are

$$x_1' = 11 > x_2' = 10.3 > x_3' = 10.1 > \ \ldots$$

Note that al-Aṣfahānī chooses the value 11 in a manner that is a little different. Instead of the function f, he considers a function g such that $f \leqslant g$, i.e.

$$g(x) = (121x)^{1/3}$$

and looks for a root x_1 of the new equation $x = g(x)$, which ensures that $x_1 = 11 > x_0$ if x_0 is the required root.

33 Sharaf al-Dīn al-Ṭūsī, vol. I, p. xxvii.

34 Al-Khilāṭī, *Nūr al-Dalāla fī ʿilm al-jabr wa al-muqābala*, MS of the University of Teheran no. 4409, folio 2.

35 See Shams al-Dīn al-Mārdīnī, *Niṣāb al-ḥabr fī ḥisāb al-jabr*, Istanbul, MS Feyzullah no. 1366, folios 13–14.